Optimal Climb Paths for a Ballistic Rocket within the Atmosphere

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Nomenclature

c $= gI =$ characteristic jet velocity, m/s
C_D = drag coefficient
D' $= \frac{1}{2}\rho_0 S C_D$, kg/m
g = acceleration due to gravity (9.81 m/s^2)
H = scale height of isothermal atmosphere (7000 m)
I = specific impulse, s
M = mass, kg
R = radius of Earth (6370×10^3 m)
s = path length, m
S = reference area of rocket for drag coefficient, m^2
t = time, s
V = velocity, m/s
V_c $= \sqrt{gR} =$ circular orbital velocity, m/s
X = horizontal distance around the Earth, m
Y = altitude above Earth surface, m
γ = climb path angle, rad
ϕ = characteristic mass $= M\exp(V/c)$, kg
ρ_0 = sea-level air density (1.25 kg/m^3)

Introduction

IN 1951, Tsien and Evans[1] successfully solved the problem of the optimal climb of a vertically ascending sounding rocket. Since then, much work, notably by Miele, has been devoted to solving the extended problem of optimal flight paths away from the vertical. Many methods of analysis have been attempted, including dynamic programming by Bellman. Reference 2 gives a summary of early work in this field.

The general problem of optimal trajectories of a winged rocket has been solved by the present author.[3]

This note considers the problem at zero incidence, which is simpler than the general case where incidence is allowed to vary. The equations of motion are deceptively simple, yet none of the methods can be said to have achieved success, or at least not with the simplicity of the method proposed in this note. It is not generally recognized that the equations of motion are integrable by quadratures, in principle, for any and all relationships of velocity vs altitude. The difficulty is to find that particular velocity vs altitude relationship which minimizes the fuel consumed due to acceleration, gravity, and drag losses, and at the same time minimizes the time to altitude, and gives a stationary value to the angle turned through by the flight path.

First, the equations of motion are derived and some preliminary analysis is carried out to show what form the simplest solution should take. The optimal solution is then derived, to show that the actual solution does take this simple form.

Equations of Motion

Several simplifying assumptions are made, but these are generally valid over the ascent path of a rocket. These are 1) spherical, nonrotating Earth, 2) constant value of gravitational force, 3) constant specific impulse I or jet efflux velocity c, 4) constant drag coefficient, 5) isothermal atmosphere with exponential variation of density with altitude, and constant scale height H, 6) thrust acts along the flight path (i.e. zero incidence and no vectored thrust). With these assumptions, the equations of motion become

$$c\frac{dM}{dt}+M\left(\frac{dV}{dt}+g\sin\gamma\right)+D'V^2\exp\left(-\frac{y}{H}\right)=0 \tag{1}$$

$$V\frac{d\gamma}{dt}+\left(g-\frac{V^2}{R}\right)\cos\gamma=0 \tag{2}$$

$$dy/dt=V\sin\gamma \tag{3}$$

$$dx/dt=V\cos\gamma \tag{4}$$

$$ds/dt=V \tag{5}$$

Use Eq. (5) to transform the other four equations to the space domain. These then become

$$\frac{dM}{ds}+M\left(\frac{1}{c}\frac{dV}{ds}+\frac{g}{Vc}\sin\gamma\right)+D'\frac{V}{c}\exp\left(-\frac{y}{H}\right)=0 \tag{6}$$

$$\frac{d\gamma}{ds}+\left(\frac{g}{V^2}-\frac{1}{R}\right)\cos\gamma=0 \tag{7}$$

$$dy/ds-\sin\gamma=0 \tag{8}$$

$$dx/ds-\cos\gamma=0 \tag{9}$$

Equation (6), regarded as an equation for M, has an integrating factor

$$\exp\left[\frac{V}{c}+\int\frac{g\,dy}{Vc}\right]=\exp\left(\frac{V}{c}+G\right)$$

where

$$G=\int_1^2\frac{g\,dy}{Vc} \tag{10}$$

Equation (7) may be written

$$-\frac{\sin\gamma}{\cos\gamma}\frac{d\gamma}{ds}=\left(\frac{g}{V^2}-\frac{1}{R}\right)\frac{dy}{ds} \tag{11}$$

and provided altitude y is a function of velocity V

$$\log(\cos\gamma)=\int_1^2\left(\frac{g}{V^2}-\frac{1}{R}\right)\frac{dy(V)}{dV}\cdot dV \tag{12}$$

giving climb angle γ as a function of velocity V.

Equation (6) becomes

$$\left[M\exp\left(\frac{V}{c}+G\right)\right]_1^2+\int_1^2\frac{D'V}{c}\exp\left(\frac{V}{c}+G-\frac{y(V)}{H}\right)\frac{dy(V)}{\sin[\gamma(V)]}=0 \tag{13}$$

Since y, G, and γ are known as functions of velocity V through the assumed velocity-altitude relationship, then Eq. (13) is integrable by quadrature for all relationships between velocity and altitude. The function G is merely the (nondimensional) gravity loss, and the integral term in Eq. (13) is the total drag loss, allowing for mass variation due to drag loss, acceleration, and gravity loss.

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Simplest Form of Analytical Solution

The above is not the simplest form that the analytical solution may take. Let

$$\tan\gamma = f(V) \tag{14}$$

Then

$$\sec^2\gamma \frac{d\gamma}{ds} = f'(V)\frac{dV}{ds}$$

Using Eq (11),

$$-\sec^2\gamma\left(\frac{g}{V^2}-\frac{1}{R}\right)\frac{\cos\gamma}{\sin\gamma}\frac{dy}{ds}$$

$$= -\left(\frac{(1+f^2)}{f}\right)\left(\frac{g}{V^2}-\frac{1}{R}\right)\frac{dy}{ds} = f'(V)\frac{dV}{ds}$$

Hence

$$\frac{1}{H}dy = -\frac{V^2}{(g-V^2/R)H}\frac{ff'}{(1+f^2)}dV \tag{15}$$

$$\frac{1}{H}dx = \frac{1}{\tan\gamma}\frac{1}{H}dy = \frac{-V^2}{(g-V^2/R)H}\frac{f'}{(1+f^2)}dV$$

$$G = \int_1^2 \frac{gH}{Vc}\frac{dy}{H} = -\int_1^2 \frac{V}{c}\frac{1}{(1-V^2/gR)}\frac{ff'}{(1+f^2)}dV$$

$$\frac{1}{H}ds = \frac{1}{\sin\gamma}\frac{dy}{H} = -\frac{V^2}{(g-V^2/R)H}\frac{f'}{\sqrt{1+f^2}}dV$$

$$\frac{g\,dt}{c} = -\frac{V}{c}\cdot\frac{1}{(1-V^2/gR)}\frac{f'}{\sqrt{1+f^2}}dV \tag{16}$$

The fuel consumption becomes

$$\left[M\exp\left(\frac{V}{c}+G\right)\right]_1^2 - \int_1^2 \frac{D'V}{c}\exp\left(\frac{V}{c}+G-\frac{y}{H}\right)$$

$$\times\frac{V^2}{(g-V^2/R)H}\frac{f'}{\sqrt{1+f^2}}dV = 0 \tag{17}$$

Thus all the equations are integrable in their simplest form provided

$$\tan\gamma = f(V)$$

Solution of Optimal Equations

The performance index to minimize is the time of flight to altitude. This can be adjoined to the equations of motion in the space domain, through use of the classical calculus of variations. It also falls naturally into the Pontryagin form.

Equations (6-9) show that there are only four differential equations between five differential variables, namely M, V, γ, y, and x. One of these differential variables is redundant, and can be eliminated by multiplying Eq. (6) by $\exp(V/c)$. Then

$$\exp\left(\frac{V}{c}\right)\frac{dM}{ds} + \frac{M}{c}\exp\left(\frac{V}{c}\right)\frac{dV}{ds} = \frac{d\phi}{ds}$$

where

$$\phi = M\exp(V/c) \tag{18}$$

Equation (6) becomes

$$\frac{d\phi}{ds} + \phi\frac{g}{Vc}\sin + \frac{D'V}{c}\exp\left(\frac{V}{c}-\frac{y}{H}\right) = 0 \tag{19}$$

The velocity V disappears as a differential variable, and reappears only as an algebraic (control) variable. This velocity V can be freely chosen to satisfy the optimal equations of ascent. The function ϕ is conserved through an impulse, is continuously and slowly dissipated along a trajectory, and represents the characteristic mass, that is, the mass M_0 before an impulse required to accelerate a mass M to velocity V. Thus in this formulation, thrust is free and unconstrained, and may require an impulsive thrust to satisfy the optimal conditions. It will be noticed that if the thrust is constrained, then so is the complete flight path, and no optimization can take place under these circumstances. Since characteristic mass ϕ gives the tradeoff between mass and velocity at the end of a trajectory, it is the characteristic mass ϕ which must be maximized at the end point. This then gives maximum mass at maximum velocity in minimum time.

Using Lagrangian undetermined multipliers (λ_i) for each of the differential variables, the variational equation for minimum time along the optimal trajectory becomes

$$\left.\frac{gt}{c}\right]_1^2 = \int_1^2\left\{\frac{g}{c}\frac{1}{V} + \lambda_1\left[\frac{d\phi}{ds} + \phi\frac{g\sin\gamma}{Vc}\right.\right.$$

$$\left.+D'\frac{V}{c}\exp\left(\frac{V}{c}-\frac{y}{H}\right)\right] + \lambda_2\left[\frac{d\gamma}{ds} + \left(\frac{g}{V^2}-\frac{1}{R}\right)\cos\gamma\right]$$

$$\left.+\frac{\lambda_3}{H}\left[\frac{dy}{ds}-\sin\gamma\right] + \frac{\lambda_4}{H}\left[\frac{dx}{ds}-\cos\gamma\right]\right\}ds = \min \tag{20}$$

All equations, except for ϕ, are nondimensionalized with respect to the appropriate scaling factor.

The Euler-Lagrange equations become

$$\frac{d}{ds}\left(\frac{\partial F}{\partial x_i'}\right) - \frac{\partial F}{\partial x_i} = 0 \tag{21}$$

for all variables $x_i = \phi, \gamma, y, x$.

With respect to ϕ

$$\frac{d}{ds}(\lambda_1) = \lambda_1\frac{g}{Vc}\sin\gamma \tag{22}$$

Hence

$$\frac{d}{ds}(\lambda_1\phi) = -\lambda_1 D'\frac{V}{c}\exp\left(\frac{V}{c}-\frac{y}{H}\right) \tag{23}$$

With respect to y

$$\frac{1}{H}\frac{d}{ds}(\lambda_3) = -\frac{1}{H}\lambda_1 D'\frac{V}{c}\exp\left(\frac{V}{c}-\frac{y}{H}\right) = \frac{1}{H}\frac{d}{ds}(\lambda_1\phi)$$

Therefore

$$\lambda_3 = (\lambda_1\phi) - C_3 \tag{24}$$

where C_3 is a constant.

With respect to x

$$\lambda_4 = C_4 \qquad \text{(constant)} \tag{25}$$

With respect to γ

$$\frac{d}{ds}(\lambda_2) = \lambda_1\frac{g}{Vc}\cos\gamma - \lambda_2\left(\frac{g}{V^2}-\frac{1}{R}\right)\sin\gamma$$

$$-\frac{\lambda_3}{H}\cos\gamma + \frac{\lambda_4}{H}\sin\gamma \tag{26}$$

Using Eq. (7), this becomes

$$\frac{d\lambda_2}{ds} - \lambda_2\frac{\sin\gamma}{\cos\gamma}\frac{d\gamma}{ds}$$

$$= -(\lambda_1\phi)\left(\frac{1}{H}-\frac{g}{Vc}\right)\cos\gamma + \frac{C_3}{H}\cos\gamma + \frac{C_4}{H}\sin\gamma$$

This has an integrating factor, $\cos\gamma$, and the equation becomes

$$\frac{d}{ds}(\lambda_2 \cos\lambda)$$

$$= \left[-(\lambda_1\phi)\frac{(Vc-gH)}{VcH}\cos\gamma + \frac{C_3}{H}\cos\gamma + \frac{C_4}{H}\sin\gamma\right]\cos\gamma$$

This is integrable if, and only if

$$(\lambda_1\phi) = \frac{Vc}{(Vc-gH)}[C_3 + C_4 \tan\gamma] \tag{27}$$

whereupon

$$\lambda_2 \cos\gamma = C_2 \qquad \text{(constant)} \tag{28}$$

With respect to the algebraic variable velocity, V

$$\lambda_1 \frac{D'}{c}\left(1+\frac{V}{c}\right)\exp\left(\frac{V}{c}-\frac{y}{H}\right) - \lambda_1\phi\frac{g}{V^2c}\sin\gamma$$

$$-\frac{2g}{V^3}(\lambda_2\cos\gamma) - \frac{g}{V^2c} = 0 \tag{29}$$

Hence

$$D'\frac{V}{c}\exp\left(\frac{V}{c}-\frac{y}{H}\right)$$

$$= \phi\frac{g}{V(c+V)}\sin\gamma + \phi\left[\frac{2gc(\lambda_2\cos\gamma)}{V^2(c+V)} + \frac{g}{V(c+V)}\right]\frac{1}{(\lambda_1\phi)} \tag{30}$$

The condition for the integral to depend only on altitude and time to altitude, and not on angle turned through, is

$$\left[\frac{2gc\cdot C_2}{V^2(c+V)} + \frac{g}{V(c+V)}\right]\frac{1}{(\lambda_1\phi)} = \frac{C_5 g}{Vc} \tag{31}$$

Hence

$$\frac{d\phi}{ds} + \phi\left[\frac{g}{Vc}\frac{dy}{ds} + \frac{g}{V(c+V)}\frac{dy}{ds} + \frac{C_5 g}{c}\frac{dt}{ds}\right]_1^2 = 0$$

or, after integration

$$\log\phi\Big]_1^2 + \int_1^2 \frac{g(2c+V)}{Vc(c+V)}dy + \left[\frac{C_5 gt}{c}\right]_1^2 = 0 \tag{32}$$

This holds provided that, from Eqs. (30) and (31)

$$D'\frac{V}{c}\exp\left(\frac{V}{c}-\frac{v}{H}\right) = \phi\left[\frac{g}{V(c+V)}\sin\gamma + \frac{C_5 g}{Vc}\right] \tag{33}$$

The similarity with Tsien and Evans' case for the vertical, $\sin\gamma = 1$ and $C_5 = 0$ can be seen immediately. The drag-to-mass variation is given by

$$D'V^2\exp\left(-\frac{y}{H}\right) = Mg\left[\frac{c}{(c+V)}\sin\gamma + C_5\right] \tag{34}$$

Also, from Eqs. (31) and (27)

$$\frac{g}{V(c+V)}\left[\frac{2c}{V}C_2 + 1\right] = C_5\frac{g}{Vc}(\lambda_1\phi)$$

$$= C_5\frac{g}{Vc}\left(\frac{Vc}{Vc-gH}\right)[C_3 + C_4\tan\gamma]$$

Hence

$$C_3 + C_4\tan\gamma = \frac{(Vc-gH)}{V(c+V)}\frac{1}{C_5}\left[\frac{V+2cC_2}{V}\right] \tag{35}$$

The term in gH/c is negligibly small, being only of the order of 28 m/s relative to velocities of hundreds or even thousands of m/s. Assuming that, in Eq. (35), $2C_2 = 1$, the terms in $(c+V)$ disappear. This assumption can only affect the angle turned through by the flight path, which can be freely chosen by choosing correctly the initial flight path angle. Thus, depending on the sign of C_3 the equations become

$$C_3 + C_4\tan\gamma = 1/C_5\cdot c/V$$

$$\tan\gamma = (a + bc/V) \tag{36}$$

Equation (36) shows that the solution of the optimal equations has the correct form for simple integration as shown by Eqs. (15) and (16). The fuel consumed follows from Eq. (32) and the drag/mass ratio from Eq. (34).

Equation (34) gives the value of C_5, the constant in the time-of-flight equation.

$$C_5 = \frac{D'V_1^2}{M_0 g}\exp\left(\frac{V_1}{c}\right) - \frac{c}{(c+V_1)}\sin(\gamma_1) \tag{37}$$

Thus C_5 can be positive, zero, or negative depending on the magnitude of the right-hand side of Eq. (37).

Conclusion

A simple and satisfactory method for finding the optimal climb path of a ballistic rocket within the atmosphere has been derived. The resulting equations are solvable by simple quadrature. The optimal solution requires an impulsive boost to start the trajectory and variable thrust to complete it, otherwise no optimization can be carried out.

References

[1]Tsien, H.S. and Evans, R.C., "Optimum Thrust Programming for a Sounding Rocket," *Journal of the American Rocket Society*, Vol. 21, Sept. 1951, pp. 99-107.

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Computational Scheme for Calculating the Plume Backflow Region

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Introduction

SEVERAL computer programs and approximate methods exist for the calculation of rocket engine exhaust plume flowfields. The majority of these computational schemes neglect the effects of the nozzle wall boundary layer on the plume flowfield and, therefore, are useful mainly for calculating the core flowfield of the plume. This restriction is usually unimportant since most of the mass contained within

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